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LETTER TO THE EDITOR

Kinetic roughening with multiplicative noise

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Abstract. In this letter a new approach to the investigation of the effects of quenched randomness in the experiments on kinetic roughening is introduced by considering a stochastic differential equation for the surface development with a multiplicative noise. We argue that this type of noise corresponds to the experimental situation in cases when the development of the interface is dominated by pinning forces. By numerically integrating the proposed equation we have obtained (i) surfaces remarkably similar to those observed in the experiments and (ii) a scaling of the surface width as a function of time with an exponent being in an excellent agreement with the measured value. Variations of the model, crossovers and questions concerning the applicability of additive noise to wetting experiments are also discussed.

During the past few years a considerable amount of interesting results has accumulated about the far from equilibrium growth of fractal surfaces [1, 2]. From the available data it is clear that perhaps the most exciting recent question regarding the growth of rough interfaces is the apparent discrepancy between the experimental results and the corresponding predictions based on the most general theoretical approaches and related simulations. In particular, for the $(1+1)$-dimensional case the renormalization group treatment of the KPZ equation [3] and large scale numerical studies of the simplest aggregation models [4] give $\alpha=\frac{1}{2}$ and $\beta=\frac{1}{3}$, where these exponents describe the scaling of the width of the surface [5]

$$w(x,t)\sim t^{\beta}f(t/x^{\alpha/\beta}) \qquad (1)$$

as a function of time t and the linear extension x of the surface over which the width is calculated.

On the other hand, the existing experimental estimates obtained for the interface of viscous flows and the surface of bacteria colonies range between 0.63 and 0.81 [6-8] for α and give $\beta\simeq 0.65$ [7]. These values are in clear conflict with the predictions $\frac{1}{2}$ and $\frac{1}{3}$. There are many more experimental systems (see, e.g., [2]) in which the measured roughness exponents differ from the KPZ values. However, in this paper we shall mainly be concentrating on the process of the advancement of a wetting fluid in inhomogeneous media [6, 7].

Recently a few specific models have been proposed to eliminate the above mentioned disagreement. Results in part consistent with the experiments have been obtained by assuming power law distributed noise amplitudes [9-13], studying a simplified KPZ type equation with quenched noise [14], by changing the growth rule in various growth models [15-18] and by modifying the equation itself [19-21] in order to better account for the physical conditions determining the process of kinetic roughening.

The purpose of the present letter is to introduce a new concept of studying the actual physical situation by considering a stochastic differential equation for the surface development with a multiplicative noise. Our goals are (i) to make assumptions which are as close to the experimental conditions as possible, (ii) to numerically investigate the resulting equation and (iii) to compare the obtained behaviour with that observed in the experiments.

We propose that the development of the interface $h(x, t)$, e.g., in the experiments on quasi-$(1+1)$-dimensional viscous flows, is described by the equation

$$\frac{\partial h}{\partial t} = (\nabla^2 h + v(1+(\nabla h)^2)^{1/2})(p+\eta) \tag{2}$$

where $p > 0$ is some constant, v is the normal velocity and the term $\eta > 0$ corresponds to quenched noise with no correlations, i.e.,

$$\langle \eta(x, h)\eta(x', h')\rangle = C\delta(x-x')\delta(h-h'). \tag{3}$$

We do not assume that the distribution of the noise amplitudes is Gaussian with a zero mean; it would be equivalent to supposing that flat parts of the interface would move backward at places with $\eta < -p$. Rather, we shall assume that η follows some other simple distribution, e.g., the Poisson or the uniform distribution. In this way, we can avoid (unlike in the case of the Gaussian distribution) the occurrence of the unphysical values $p+\eta < 0$.

To support the particular form in which the noise term enters (2) let us consider the experiment on the two-phase flow of viscous fluids in porous media. We are interested in the case when the more viscous, wetting fluid advances due to the presence of capillary forces and the interface exhibits kinetic roughening. Under such circumstances the system can be considered as a network of randomly interconnected channels of widely distributed sizes and geometry. The motion of the wetting fluid is determined by the simultaneous effects of surface tension, capillary forces and local flow properties (permeabilities of the channels). The advancement of the interface at a given point is proportional to the local driving force and the permeability (Darcy's law), just as the electric current j is proportional to the conductivity σ and the electric field $\boldsymbol{E}$, $\boldsymbol{j} = \boldsymbol{E}\sigma$. In our case the driving forces are (i) the wetting or capillary force which would produce velocity v for unit permeability and (ii) the forces due to the surface tension which are represented by the term $\nabla^2 h$ (we assume that there is no extra pressure applied to the penetrating fluid). Thus, (2) is equivalent to

$$v_s = F\varepsilon \tag{4}$$

where v_s is the velocity of the surface in the vertical direction, ε is the randomly changing local permeability and F denotes a general driving force. Although here we used the wetting experiment as an example to justify the necessity to take into account multiplicative noise, we think that in many other situations (e.g., motion of domain walls in magnetic systems with random fields and pinning of charge density waves) our approach should also be considered.

Next we would like to make a few relevant comments on the other aspects of the proposed equation. (i) The term $v(1+(\nabla h)^2)^{1/2}$ is included in its full form (instead of its linearized version used in the KPZ equation), because in the actual experiments at the majority of the points along the interface the condition $|\nabla h| \ll 1$ is not satisfied. This statement becomes very relevant when pinning forces are present and the interface develops deep valleys with $|\nabla h| \gg 1$ playing a determining role in the process of roughening. (ii) Naturally, (2) can be extended by including other terms, e.g., an explicit additive noise ζ which can be independent of or proportional to η, and a term $\lambda(\nabla h)^2$ instead of $(\nabla h)^2$ only (λ is a parameter). In this case (2) reads as

$$\frac{\partial h}{\partial t}=(\nabla^2 h+v(1+\lambda(\nabla h)^2)^{1/2})(p+\eta)+\zeta. \tag{5}$$

In fact, even without including additive noise explicitly, in (2) or in the above equation the term $v(1+\lambda(\nabla h)^2)^{1/2}\eta$ has a contrbution which corresponds to additive noise. If both types of noises are present, one may expect that in the limit of very large system sizes and long times the additive noise will dominate the growth since the various derivatives of the surface become very small on a coarse grained scale. The detailed discussion of this important question cannot be included in the present work and should be addressed separately. The only feature we note here is that choosing λ, $v \ll 1$ in (3) the additive noise can be made arbitrarily small and the behaviour is determined by the multiplicative nature of the noise. This regime then is likely to cross over to the additive noise-dominated case after arbitrarily long characteristic time.

Now we are in the position to describe the development of the interface in terms of kinetic roughening dominated by pinning forces. At places where $p+\eta \ll 1$, the motion of the interface slows down dramatically. These points can be considered as temporarily pinned. However, as in all existing experiments on growth (with no evaporation), after some time the surface passes by this place or region of low permeability and advances further without a complete stop.

An interesting special case of the noise is when η depends only on x (this possibility is also discussed in [15], for additive noise). An existing aspect of the physics is reflected by this choice: the motion of the interface is determined not only by the conditions at the surface, but also by the permeability of regions already left behind (which may partially block the supply of additional fluid). A possible realization of this case includes a Hele-Shaw cell with parallel grooves of different depth engraved onto one of the glass plates.

Before describing our numerical studies of (2) we briefly discuss the *applicability* of the KPZ approach to the experiments on wetting fronts. According to the KPZ equation the development of the surface is described by

$$\frac{\partial h}{\partial t}=\nabla^2 h+\lambda/2(\nabla h)^2+v+\eta(x,t) \tag{6}$$

where λ is a parameter which for wetting flows is larger than 0. As was pointed out by Kessler *et al* [14], for the interpretation of the experiments it is more appropriate to use a quenched noise in (6), $\eta(x,h)$, and this is the version we shall discuss below.

At the places where the surface is locally almost pinned (slowed down) $\partial h/\partial t \ll 1$. On the other hand, at the same locations $\nabla^2 h+\lambda/2(\nabla h)^2 \gg 1$. According to (6) this can hold only if $-(v+\eta)\gg 1$ in these points. We argue that large negative values of the noise η are not physical, because this would mean that a flat surface in the given point would move with a large velocity in the direction *opposite* to the growth. Since the fluid is wetting, its spontaneous motion cannot be reverse.

Nevertheless, the equation (6) with quenched noise can be solved numerically making various assumptions for η. Without mentioning the details, we would like to point out that such an approach does not lead to surfaces similar to the experimental ones (e.g., in figure 2 of [14] the axes are not isotropic; if the scales along the axes were the same (as in the experimental pictures) the calculated surface would look almost like a straight line).

Since the main goal of this work is to understand what are the most relevant factors determining the behaviour of experimental surfaces, we have numerically studied (2) for times and system sizes compatible to those which have been realized in [6] and [7]. It is straightforward to integrate (2) numerically; the associated questions are discussed in [14] (quenched case) and [22] (for time dependent noise). For simplicity we assumed that η was distributed uniformly on (0, 1]. Figure 1 shows the calculated surfaces for the following set of parameters: system size $L = 800$, $p = 0.0001$ and $v = 0.5$. It should be pointed out that in the present case the mesh size used in the course of discretizing the x dependence in (2) has a physical meaning: it corresponds to the lower cutoff length scale of the fluctuations of the media (for example, it can be identified with the diameter of the glass beads in the experiments of [6] and [7], since η is assigned to the grid points of a square lattice with mesh size Δx). In t a finer discretization is used, and the actual value of h is a quasi-continuous variable.

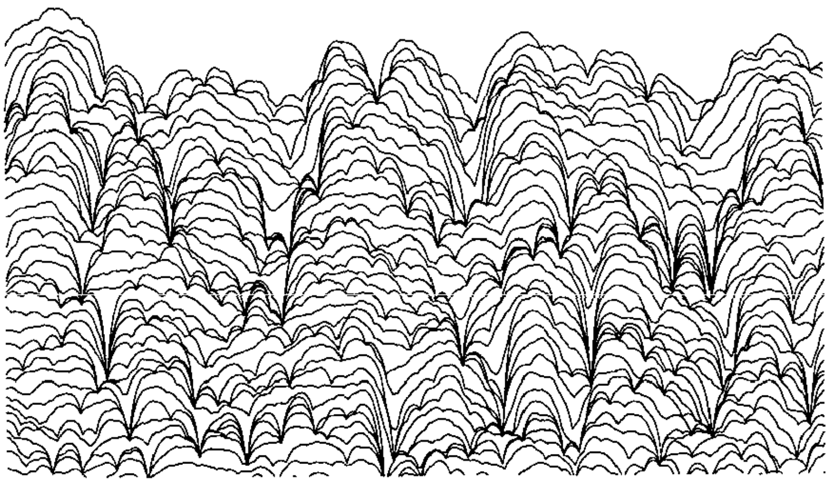

Figure 1. Subsequent 'snapshots' of the evolving surface obtained by numerically integrating (2) for $p = 0.0001$ and $v = 0.5$ for a system of linear size $L = 800$.

Next we investigated $w(L, t)$, the time dependence of the width of the entire system. The results shown in figure 2 indicate that at early times there exists a non-trivial scaling

$$w \sim t^{\beta} \tag{7}$$

with $\beta \simeq 0.65$ in surprisingly good agreement with the only published experimental result. This value also agrees with the corresponding estimates obtained in [16] and [17], but is different from $\frac{3}{4}$ published in [15]. The crossover to a behaviour described by a smaller exponent $\beta = 0.28 \pm 0.05$ is well pronounced and according to our simulations the crossover time t_c does not depend on L and roughly scales as a function of p as $p^{0.7}$. Our preliminary calculations indicate that for larger system sizes and longer times the value 0.28 tends to increase and we expect it to approach $\frac{1}{3}$ in the asymptotic limit. This limit can be reliably investigated only by using supercomputers, and the corresponding studies will be reported later.

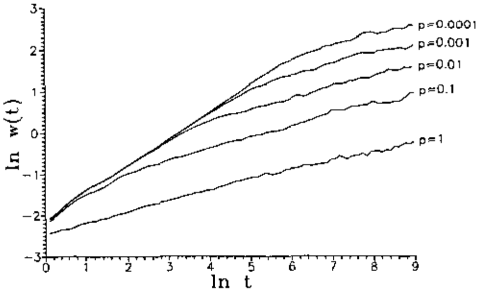

Figure 2. The time dependence of the surface width w for various values of the parameter p related to the strength of the pinning forces (smaller p corresponds to stronger pinning). There is a well defined crossover at a time t_c, depending on p, from a scaling according to an exponent $\beta \simeq 0.65$ to a scaling with β about 0.28.

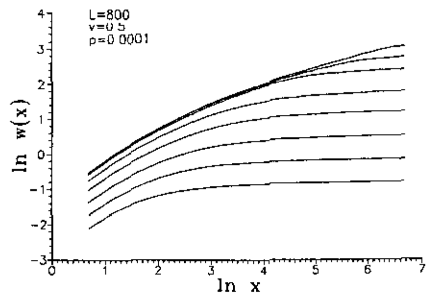

Figure 3. The behaviour of the surface width w as a function of the length x over which its average was calculated. This sequence of curves was obtained for a selection of increasing elapsed times (from bottom to top). The slope of the topmost curve for large x values is close to 0.5.

The spatial scaling of w also raises interesting questions. According to our calculations there exist no well defined, extended straight parts in the log w versus log x plots (see figure 3). On the other hand, the curves can be approximately described in terms of an initial larger slope of about 0.7 crossing over to a behaviour corresponding to a surface having a roughness exponent equal to the universal value $\frac{1}{2}$.

Finally, we briefly discuss the case when η depends on x only. Figure 4 shows a typical series of surfaces for $L = 800$, $p = 0.0001$ and $v = 0.5$. In this model there is well pronounced scaling both in time and space, with numerically determined exponents close to 1 ($\alpha \simeq \beta \simeq 0.96 \pm 0.06$).

In conclusion, we have proposed an approach which is intended to take into account the experimental conditions during two-phase fluid flows in inhomogeneous media as

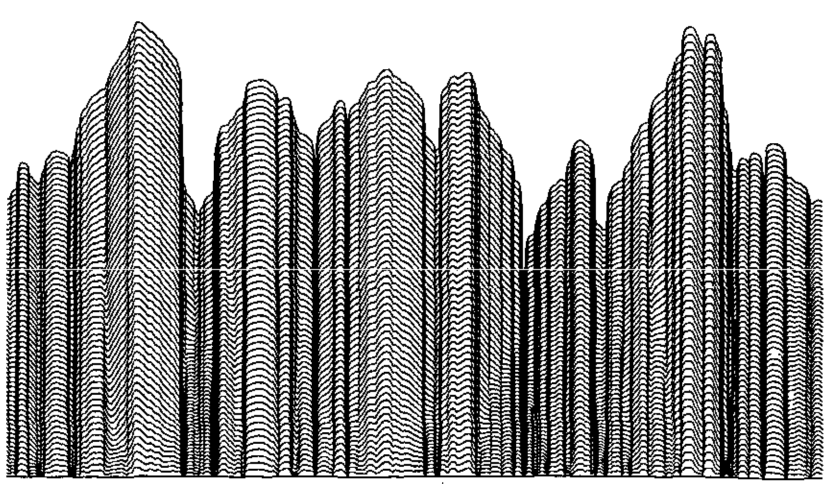

Figure 4. Series of surfaces obtained as a function of time for the model with η depending on x only (see the text).

well as possible. By introducing a stochastic partial differential equation with a multiplicative noise describing the development of the interface we have been able to obtain (i) surfaces remarkably similar to those observed in the experiments and (ii) a scaling behaviour of the surface width with an exponent being in an excellent agreement with the measured value.

There are three recent models designed to simulate growth in the presence of pinning forces. In the very interesting approach by Parisi [15] the noise is additive, and the questions raised in this paper have to be considered when discussing its applicability to wetting experiments. The growth models proposed in [16] and [17] are somewhat closer to the approach introduced in this paper. For example, the blocking sites of [16] correspond to places with very small $p+\eta$. However, the actual mechanisms of growth and in the cluster growth and the present pictures are rather different.

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